TRINITY COLLEGE

IN NOMINE DOMINI

Semester Two Examination, 2019

Question/Answer booklet

MATHEMATICS SPECIALIST UNITS 1 AND 2

Section Two: Calculator-assumed

SOLUTIONS

Student number: In figures

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	55	35
Section Two: Calculator-assumed	13	13	100	99	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answer to the specific question asked and to follow any instructions that are specified to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **65% (99 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(7 marks)**

Let S be the set of integers between 1 and 79 inclusive.

(a) Determine how many of the integers in S are

(i) a multiple of 4. (1 mark)

Solution
$\lfloor 79 \div 4 \rfloor = 19$
Specific behaviours
✓ correct number

(ii) a multiple of 4 or a multiple of 3. (3 marks)

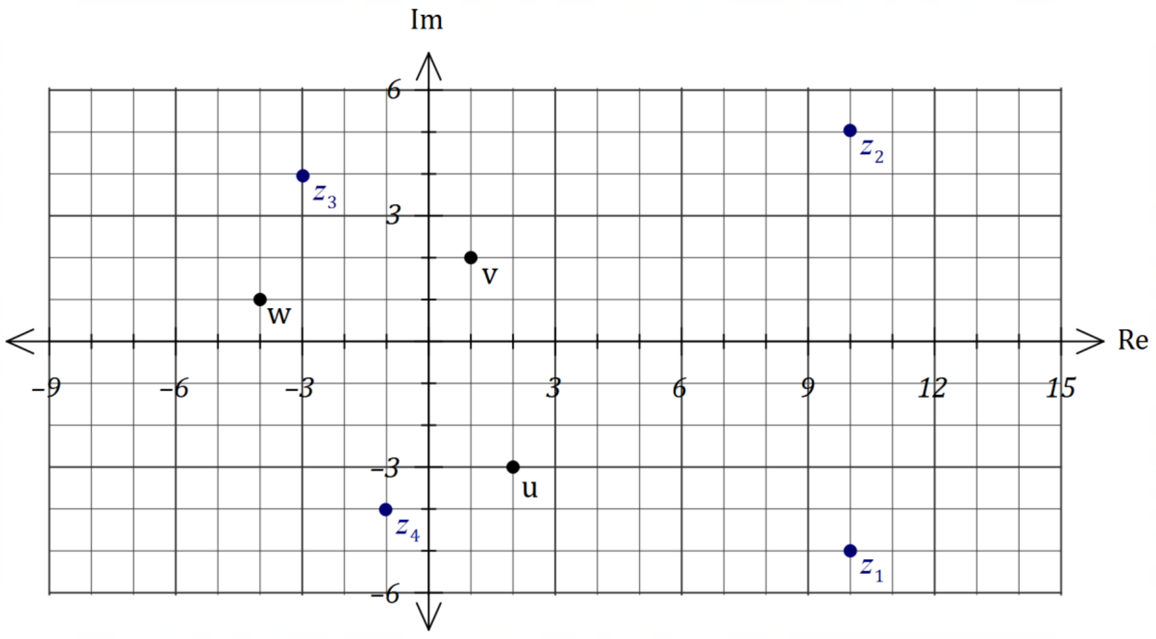
Solution
$\lfloor 79 \div 3 \rfloor = 26$ $\lfloor 79 \div (3 \times 4) \rfloor = 6$ $n = 19 + 26 - 6 = 39$
Specific behaviours
✓ multiples of 3 and multiples of 12 ✓ uses inclusion-exclusion principal ✓ correct number

(b) Integers are selected one at a time, at random and without replacement from S. After how many selections can you be certain that the squares of at least three of the integers selected will share the same last digit? Justify your answer. (3 marks)

Solution
Squares of numbers in S will end in $0, 1, 4, 5, 6, 9$ and so there are 6 pigeon-holes. Select $6 \times 2 + 1 = 13$ to be certain.
Specific behaviours
✓ identifies possible last digits ✓ indicates use of pigeon-hole principle ✓ correct number

Question 10 **(7 marks)**

The location of u, v and w in the complex plane are shown below.



(a) Plot and label the following on the same diagram:

(i) $z_1 = u - 2w$. (2 mark)

(ii) $z_4 = wi$. (2 mark)

Solution
$z_1 = 2 - 3i - 2(-4 + i) = 10 - 5i$
$z_4 = (-4 + i)i = -1 - 4i$
Specific behaviours
✓✓✓✓ each correctly plotted point

(b) The complex number v is a solution to the equation $z^2 + az + b = 0$. Determine the value of the real constant a and the real constant b. (3 marks)

Solution
$v = 1 + 2i, \qquad \bar{v} = 1 - 2i$
$\big(z - (1 + 2i)\big)\big(z - (1 - 2i)\big) = 0$ $z^2 - 2z + 5 = 0$
$a = -2, \qquad b = 5$
Specific behaviours
✓ identifies $\bar{v}$ as second solution   ✓ forms factors   ✓ expands and states values

Alternative solution
$v = 1 + 2i, \qquad \bar{v} = 1 - 2i$
$a = -(v + \bar{v}) = -2$
$b = v \times \bar{v} = 5$
Specific behaviours
✓ identifies $\bar{v}$ as second solution   ✓ negates sum of roots   ✓ product of roots

Question 11 **(8 marks)**

Quadrilateral $PQRS$ has vertices $P(3,-2), Q(2,3), R(7,2)$ and $S(10,-5)$.

(a) Show that $\overrightarrow{PR}$ is perpendicular to $\overrightarrow{QS}$. (3 marks)

Solution
$\overrightarrow{PR} = \begin{pmatrix}7\\2\end{pmatrix} - \begin{pmatrix}3\\-2\end{pmatrix} = \begin{pmatrix}4\\4\end{pmatrix}$ $\overrightarrow{QS} = \begin{pmatrix}10\\-5\end{pmatrix} - \begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}8\\-8\end{pmatrix}$ $\overrightarrow{PR} \cdot \overrightarrow{QS} = (4 \times 8) + (4 \times -8) = 32 - 32 = 0$ Hence $\overrightarrow{PR}$ is perpendicular to $\overrightarrow{QS}$.
Specific behaviours
✓ first vector   ✓ second vector   ✓ forms and interprets dot product

(b) Show that $|\overrightarrow{QS}| < |\overrightarrow{QR}| + |\overrightarrow{RS}|$. (3 marks)

Solution
$\lvert\overrightarrow{QS}\rvert = 8\sqrt{2} \approx 11.3$ $\lvert\overrightarrow{QR}\rvert = \left\lvert\begin{pmatrix}5\\-1\end{pmatrix}\right\rvert = \sqrt{26} \approx 5.1$ $\lvert\overrightarrow{RS}\rvert = \left\lvert\begin{pmatrix}3\\-7\end{pmatrix}\right\rvert = \sqrt{58} \approx 7.6$ $\lvert\overrightarrow{QR}\rvert + \lvert\overrightarrow{RS}\rvert = 12.7$ Hence $\lvert\overrightarrow{QS}\rvert < \lvert\overrightarrow{QR}\rvert + \lvert\overrightarrow{RS}\rvert$ since $11.3 < 12.7$
Specific behaviours
✓ correct vectors   ✓ correct magnitudes   ✓ shows true

(c) Determine the angle between $\overrightarrow{QR}$ and $\overrightarrow{RS}$ to the nearest degree. (2 marks)

Solution
$\cos\theta = \dfrac{\begin{pmatrix}5\\-1\end{pmatrix} \cdot \begin{pmatrix}3\\-7\end{pmatrix}}{\sqrt{26}\sqrt{58}}$ $\theta = 55.49 \approx 55°$
Specific behaviours
✓ method   ✓ correct angle

Question 12 **(8 marks)**

Triangle ABC is transformed by matrix $T = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$ to $A'B'C'$ and then rotated 90° clockwise about the origin to $A''B''C''$.

The coordinates of A and B are $(-3, 5)$ and $(4, 7)$ respectively and the area of ABC is 35.5 square units.

(a) Determine the coordinates of A'. (1 mark)

Solution
$\begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} -9 \\ -20 \end{bmatrix}$ $A'(-9, -20)$
Specific behaviours
✓ correct coordinates

(b) Determine matrix S that represents a 90° clockwise rotation about the origin. (1 mark)

Solution
$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
Specific behaviours
✓ correct matrix

(c) Determine the coordinates of B''. (2 marks)

Solution
$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} -28 \\ -12 \end{bmatrix}$ $B''(-28, -12)$
Specific behaviours
✓ indicates product STB ✓ correct coordinates

(d) The coordinates of C'' are $(16, -3)$. Determine

(i) the coordinates of C. (2 marks)

Solution
$\begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}^{-1} \times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} \times \begin{bmatrix} 16 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$ $C(1, -4)$
Specific behaviours
✓ indicates valid product such as $T^{-1}S^{-1}C''$ ✓ correct coordinates

(ii) the area of triangle $A''B''C''$. (2 mark)

Solution
$35.5 \times \lvert ST \rvert = 35.5 \times 12 = 426$ sq units
Specific behaviours
✓ correct area

Question 13 **(7 marks)**

The air pressure in a tank can be modelled by the equation

$$p = a + b\cos\big(c(t+d)\big) \text{ for } 0 \le t \le 24$$

where p is the pressure in kPa, t is the time in hours after midnight and all other variables are positive constants.

The air pressure first reached a minimum of 92 kPa when $t = 0.5$ h and then rose during the next 3 hours to a maximum of 116 kPa before decreasing again.

(a) Determine the value of each of the positive constants a, b, c and d. (4 marks)

Solution
$\text{Amplitude} = b = \frac{1}{2}(116 - 92) = 12$ $a = b + 92 = 12 + 92 = 104$ $\text{Period} = 2 \times 3 = 6 \Rightarrow c = \frac{2\pi}{6} = \frac{\pi}{3}$ $\cos\left(\frac{\pi}{3}(0.5 + d)\right) = -1 \Rightarrow d = 2.5$
Specific behaviours
✓✓✓✓ each correct value

(b) Use the model to determine

(i) the air pressure at 6 pm. (1 mark)

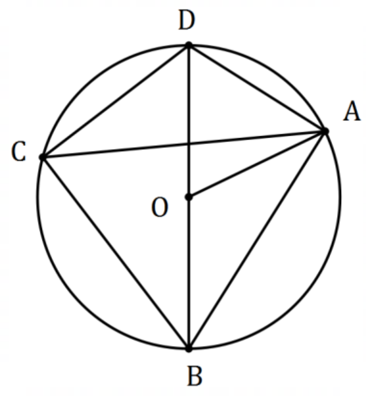
Solution
$p(18) = 104 - 6\sqrt{3} \approx 93.6 \text{ kPa}$
Specific behaviours
✓ correct pressure

(ii) the time of day, to the nearest minute, that the pressure first reached 111 kPa. (2 marks)

Solution
$p(t) = 111 \Rightarrow t = 2.595$ At $2{:}36$ am.
Specific behaviours
✓ solves for t ✓ time to nearest minute

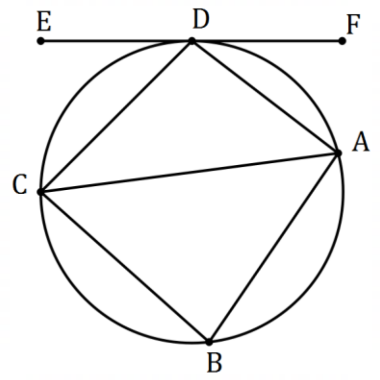
Question 14 **(9 marks)**

(a) In the diagram below, $A, B\ C$ and D lie on the circle with centre O. If $\angle BDC = 54°$ and $\angle ACB = 48°$, determine with reasoning $\angle DAC$ and $\angle AOD$. (4 marks)



Solution
$\angle DBC = 90° - \angle BDC$ (Angle in semicircle) $= 90° - 54° = 36°$ $\angle DAC = \angle DBC$ (Stand on same arc) $= 36°$ $\angle AOB = 2 \times \angle ACB$ (Angle at centre) $= 2 \times 48° = 96°$ $\angle AOD = 180° - \angle AOB$ (Angle on line) $= 180° - 96° = 84°$
Specific behaviours
✓ correct value for $\angle DAC$ ✓ reasoning to obtain $\angle DAC$ ✓ correct value for $\angle AOB$ ✓ reasoning to obtain $\angle AOB$

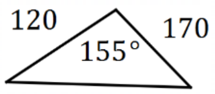
(b) In the diagram below, $ABCD$ is a cyclic quadrilateral and EF is a tangent to the circle at D. If $\angle BAC = 50°$, $\angle ADF = 38°$ and $\angle ADC = 96°$ prove that AD is parallel to BC. (5 marks)



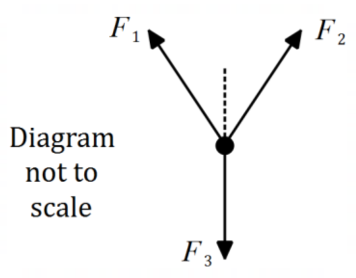
Solution
$\angle ABC = 180° - \angle ADC$ (Cyclic quadrilateral) $= 180° - 96° = 84°$ $\angle BCA = 180° - \angle ABC - \angle BAC$ (Triangle) $= 180° - 84° - 50° = 46°$ $\angle CDE = 180° - \angle ADC - \angle ADF$ (Angle on line) $= 180° - 96° - 38° = 46°$ $\angle CAD = \angle CDE$ (Angle in opposite segment) $= 46°$ Hence AD is parallel to BC as alternate angles $\angle BCA$ and $\angle CAD$ are equal.
Specific behaviours
✓ correct value for $\angle BCA$ ✓ reasoning to obtain $\angle BCA$ ✓ correct value for $\angle CAD$ ✓ reasoning to obtain $\angle CAD$ ✓ states parallel with reasons

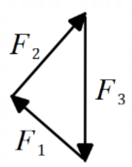
Question 15 **(7 marks)**

(a) Two forces of magnitudes 120 N and 170 N are such that one is inclined at 25° to the other. Determine the magnitude of their resultant. (3 marks)

Solution
 $r^2 = 120^2 + 170^2 - 2(120)(170)\cos 155$ $r = 283$ N
Specific behaviours
✓ diagram ✓ indicates use of cosine rule ✓ correct resultant

(b) A weight is in equilibrium, suspended by two ropes. The diagram below shows forces F_1 and F_2 acting in the ropes and F_3 exerted by the weight. If F_1 has a magnitude of 20 N and acts upwards at an angle of 55° to the vertical and F_2 acts upwards at an angle of 15° to the vertical, determine the magnitude of F_2 and F_3. (4 marks)



Solution
 $\dfrac{F_2}{\sin 55} = \dfrac{20}{\sin 15} \Rightarrow F_2 = 63.3$ N $\dfrac{F_3}{\sin 110} = \dfrac{20}{\sin 15} \Rightarrow F_3 = 72.6$ N
Specific behaviours
✓ draws vector triangle ✓ equation for F_2 ✓ solves for F_2 ✓ solves for F_3

Question 16 **(7 marks)**

A system of equations is given by

$$3x - 2y + 6 = 0$$
$$-6x + ay - 18 = 0$$

(a) Let the constant $a = 5$.

(i) Express the system in matrix form $AX = B$, where X and B are column matrices. (2 marks)

Solution
$\begin{bmatrix} 3 & -2 \\ -6 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ 18 \end{bmatrix}$
Specific behaviours
✓ matrix A
✓ correct matrix equation

(ii) Determine A^{-1} and demonstrate use of matrix algebra to solve the system for X. (3 marks)

Solution
$A^{-1} = \frac{1}{3}\begin{bmatrix} 5 & 2 \\ 6 & 3 \end{bmatrix}$ $X = A^{-1}B$ $= \frac{1}{3}\begin{bmatrix} 5 & 2 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} -6 \\ 18 \end{bmatrix}$ $= \begin{bmatrix} 2 \\ 6 \end{bmatrix}$
Specific behaviours
✓ inverse of A
✓ writes $X = A^{-1}B$
✓ correct solution

(b) Determine the value of a for which the system has no solution and comment on the relationship between the two lines that form the system when a has this value. (2 marks)

Solution
$\lvert A\rvert = 0 \Rightarrow 3a - 12 = 0 \Rightarrow a = 4$ The lines are parallel but have different y-intercepts and so do not intersect.
Specific behaviours
✓ value of a
✓ lines do not intersect

Question 17 **(7 marks)**

(a) Show that the sum of the recurring decimals $0.\overline{16} + 0.\overline{351}$ is a rational number. (3 marks)

Solution	
$x = 0.\overline{16}$ $100x = 16.161616 \ldots$ $x = 0.161616 \ldots$ $99x = 16$ $x = \dfrac{16}{99}$	$y = 0.\overline{351}$ $100y = 351.351351 \ldots$ $y = 0.351351 \ldots$ $999y = 351$ $y = \dfrac{351}{999} = \dfrac{13}{37}$
$x + y = \dfrac{16}{99} + \dfrac{13}{37} = \dfrac{1879}{3663}$	
Specific behaviours	
✓ expresses one number as rational ✓ expresses both in rational form ✓ correct sum as rational	

(b) Use algebraic reasoning to prove that if m is one more than a multiple of 3, then $m^2 + 2$ will always be a multiple of 3. (4 marks)

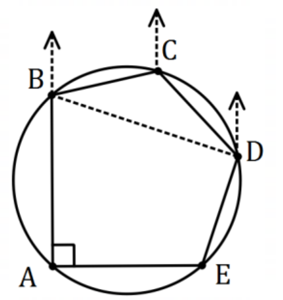
Solution
$m = 3n + 1, n \in \mathbb{Z}$ $m^2 + 2 = (3n+1)^2 + 2$ $= 9n^2 + 6n + 3$ $= 3(3n^2 + 2n + 1)$ Hence $m^2 + 2$ will always be a multiple of 3.
Specific behaviours
✓ expresses $m = 3n + 1$ ✓ correct expansion for m^2 ✓ expresses $m^2 + 2$ as $3(f(n))$ ✓ concludes as required

Question 18 **(6 marks)**

The points A, B, C, D and E lie in that order on a circle marked out on a level school playing field. A is due west of E and B is due north of A. The bearing of B from D is $305°$ and the bearing of D from C is $148°$.

(a) Sketch a diagram to show the approximate positions of A, B, C, D and E. (1 mark)

Solution



Specific behaviours

✓ circle with points in order

(b) Explain why $\angle BDE$ is a right angle. (1 mark)

Solution

$\angle BAE = 90°$ (Given)

$\angle BDE = 180° - \angle BAE = 90°$ (ABDE cyclic quadrilateral)

Specific behaviours

✓ uses cyclic quadrilateral (or other correct method)

(c) Determine the bearing of

(i) E from D. (2 marks)

Solution

Acute angle between BD and north is $360° - 305° = 55°$ and $\angle BDE = 90°$.

Hence bearing is $360° - 55° - 90° = 215°$.

Specific behaviours

✓ indicates suitable method

✓ correct bearing

(ii) A from C. (2 marks)

Solution

Acute angle between CD and north is $180° - 148° = 32°$.

$$\angle BDC = 55° - 32° = 23°$$

$$\angle BAC = \angle BDC = 23° \text{ (Same arc)}$$

Bearing is $180° + 23° = 203°$.

Specific behaviours

✓ indicates $\angle BDC$

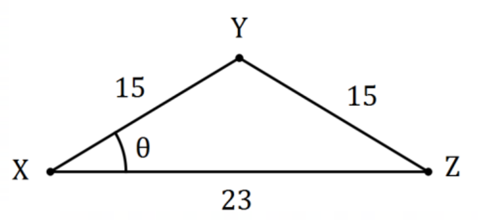
✓ correct bearing

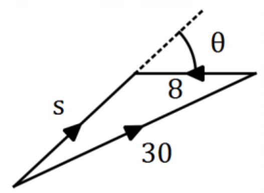
Question 19 **(8 marks)**

A boat that travels at 30 km/h in still water starts from the corner X of an isosceles triangle XYZ where $XY = YZ = 15$ km and $XZ = 23$ km and describes the complete course $XYZX$ in the least possible time. A current of 8 km/h runs in the direction $\overrightarrow{ZX}$.

(a) Determine the speed of the boat between X and Y. (5 marks)

Solution





$$\theta = \cos^{-1}\left(\frac{11.5}{15}\right) = 39.9°$$

$$30^2 = s^2 + 8^2 - 2(s)(8)\cos 140.1°$$

$$s = 23.42 \text{ km/h}$$

Specific behaviours

✓ diagram of course
✓ solves for θ
✓ diagram to add velocities
✓ equation for s
✓ correct speed

(b) Determine the time the boat takes to complete the course to the nearest minute. (3 marks)

Solution

$$t_{XY} = 15 \div 23.42 = 0.64 \text{ h}$$
$$t_{YZ} = t_{XY}$$
$$t_{ZX} = 23 \div 38 = 0.61 \text{ h}$$

$$\begin{aligned} t_{XYZX} &= 2(0.64) + 0.61 \\ &= 1.89 \\ &= 1 \text{ h } 53 \text{ m} \end{aligned}$$

Specific behaviours

✓ times for XY and YZ
✓ time for ZX
✓ correct time, to nearest minute

Question 20 **(9 marks)**

Nine congruent cubes, each of a different colour, are to be arranged in a straight line. One of the cubes is red and another is blue.

(a) Determine how many different arrangements of all 9 cubes are possible if

(i) there are no restrictions. (1 mark)

Solution
$9! = 362\ 880$
Specific behaviours
✓ correct number

(ii) the red cube must not be next to the blue cube. (3 marks)

Solution
Red next to blue: $2!$ ways, arrange 8 objects: $8!$ Number of ways adjacent is $8! \times 2! = 80\ 640$ Number not adjacent: $362880 - 80640 = 282\ 240$
Specific behaviours
✓ arranges 8 objects ✓ correct number adjacent ✓ correct number apart

(b) Determine how many different arrangements of 3 cubes chosen from the 9 are possible if

(i) there are no restrictions. (1 mark)

Solution
${}^9P_3 = 504$
Specific behaviours
✓ correct number

(ii) the arrangement must include the red cube. (2 marks)

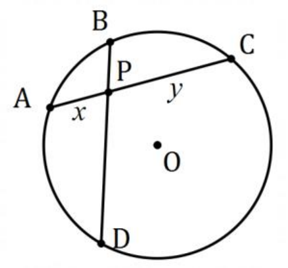
Solution
Choose red: ${}^1C_1$ and two others: ${}^8C_2$ and arrange: ${}^1C_1 \times {}^8C_2 \times 3! = 168$
Specific behaviours
✓ appropriate method ✓ correct number

(iii) the arrangement must not have the red cube next to the blue cube. (2 marks)

Solution
Choose red & blue: ${}^2C_2$ and one other: ${}^7C_1$ and arrange together: ${}^2C_2 \times {}^7C_1 \times 2! \times 2! = 28$ Number not adjacent: $504 - 28 = 476$
Specific behaviours
✓ appropriate method ✓ correct number

Question 21 **(9 marks)**

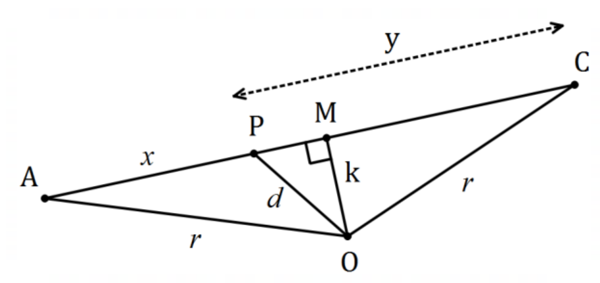
Points A, B, C and D lie on a circle such that chords AC and BD intersect inside the circle at P as shown. Let $AP = x$ and $CP = y$.



M is the midpoint of AC and the circle has radius r and centre O so that $OP = d$ and $OM = k$.

(a) Sketch triangle OAC to show this information. (2 marks)

Solution



Specific behaviours

✓ triangle with OP and OM
✓ all labels (*right-angle not necessary*)

(b) Given that $AM = \frac{x+y}{2}$, show that $PM = \frac{y-x}{2}$. (2 marks)

Solution

$$AM = \frac{x+y}{2}$$

$$\begin{aligned} PM &= AM - x \\ &= \frac{x+y}{2} - x \\ &= \frac{y-x}{2} \end{aligned}$$

Specific behaviours

✓ length AM
✓ length PM

(c) Use triangle OAM to write a relationship between k, r, x and y, and use triangle OPM to write a relationship between k, d, x and y. (2 marks)

Solution
Triangles are right-angled. $$k^2 + \left(\frac{x+y}{2}\right)^2 = r^2$$ $$k^2 + \left(\frac{y-x}{2}\right)^2 = d^2$$
Specific behaviours
✓ relationship with r ✓ relationship with d

(d) Show that $r^2 - d^2 = AP \times PC$. (2 marks)

Solution
Subtracting equations from (d) to eliminate k $$\begin{aligned} r^2 - d^2 &= \left(\frac{x+y}{2}\right)^2 - \left(\frac{y-x}{2}\right)^2 \\ &= \left(\frac{x+y}{2} - \frac{y-x}{2}\right)\left(\frac{x+y}{2} + \frac{y-x}{2}\right) \\ &= xy \\ &= AP \times PC \end{aligned}$$
Specific behaviours
✓ eliminates k ✓ shows steps to simplify expression

(e) If the radius of the circle is 65 cm, $BD = 110$ cm and $BP = 11$ cm, determine the distance of P from the centre of the circle. (1 mark)

Solution
$$\begin{aligned} 65^2 - d^2 &= xy \\ &= AP \times PC \\ &= BP \times PD \\ &= 11(110 - 11) \\ d &= 56 \text{ cm} \end{aligned}$$
Specific behaviours
✓ correct distance

End of questions

Supplementary page

Question number: _________

Supplementary page

Question number: _________